Concurrence and Foliations Induced by Some 1-Qubit Channels

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We start with a short introduction to the roof concept. An elementary discussion of phase-damping channels shows the role of antilinear operators in representing their concurrences. A general expression for some concurrences is derived. We apply it to 1-qubit channels of length 2, getting the induced foliations of the state space, the optimal decompositions, and the entropy of a state with respect to these channels. For amplitude-damping channels one obtains an expression for the Holevo capacity allowing for easy numerical calculations.

KEY WORDS: quantum channels; entropy; entanglement; roof extensions.

1. INTRODUCTION

The aim of this paper is to calculate some entanglement quantifying functions for a rather restricted class of quantum channels, in particular for all 1-qubit channels of length 2. These channels are now well examined and classified because of the work Fujiwara and Algoet (1998), King and Ruskai (2001), Ruskai *et al.* (2001), and Verstraete and Verschelde (2002). They include all external 1-qubit channels and some important doubly stochastic ones. An introduction to quantum channels is in Nielsen and Chuang (2000).

A channel, say T, is of rank 2, if $T(\rho)$ is of rank 2 for all density operators. In this case there are at most two eigenvalues of $T(\rho)$ different from zero. It follows from trace preserving that $T(\rho)$ is characterized by its largest eigenvalue up to a general unitary transformation. Nevertheless, the treatment of a general rank 2 channel is beyond our present abilities if one asks for quantities like capacity or concurrence.

A completely positive map is of length 2, if it can be written down with two Kraus operators, but not with one.

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Let us now repeat some definitions and properties of the quantities to be treated. An *ensemble* is a finite number of density operators, every one given with a definite probability or weight,

$$\mathcal{E} = \{\rho_1, \ldots, \rho_m; p_1, \ldots, p_m\}, \tag{1}$$

such that the sum of the positive numbers p_j is 1. The density operators are *states* of a physical system. We may think of a *quantum alphabet* with *quantum letters* ρ_j, and of a *quantum message* which chooses randomly the quantum letter ρ_j with probability p_j.

The weighted sum of (1) is the average density of the ensemble,

$$\mathrm{av}[\mathcal{E}] := \sum p_j \rho_j. \tag{2}$$

While the quantum letters of $\mathcal{E}$ are states of a physical system, the average (2) is a "noncommutative probability measure." Its role is quite similar to a "classical" discrete probability distribution. The extractable information per quantum letter of the quantum message is bounded by Holevo's (Holevo, 1973)

$$\chi(\mathcal{E}) := S(\rho) - \sum p_j S(\rho_j) = \sum p_j S(\rho \| \rho_j), \qquad \rho = \mathrm{av}[\mathcal{E}]. \tag{3}$$

Here $S(.)$ is the von Neumann entropy and $S(.\|.)$ the relative entropy.

Remark 1. The important inequality

$$\chi(T(\mathcal{E})) \leq \chi(\mathcal{E})$$

follows from the monotonicity of relative entropy which can be proved for trace preserving, at least two-positive maps. Counter examples for just positive maps seem to be unknown. There are two recent reviews on the finer properties of relative entropy, one by Petz (2002), going beyond complete positivity, and one by Ruskai (2002).

Let T be a positive trace preserving map acting on the density operators of $\mathcal{H}$. The application of $p \to T(\rho)$ generates, letter by letter, a new quantum message belonging to the new ensemble

$$T(\mathcal{E}) = \{T(\rho_1), \ldots, T(\rho_m); p_1, \ldots, p_m\}. \tag{4}$$

The *1-shot* or *Holevo capacity* of a channel T is the number

$$\mathrm{C}^{(1)}(\mathrm{T}) := \max_{\mathcal{E}} \chi(\mathrm{T}(\mathcal{E})). \tag{5}$$

We are aiming at a slightly more delicate expression. To see its significance, we try to perform (5) in two steps. At first we let $\mathcal{E}$ run only through the ensembles with a given average density ρ, postponing the search for the maximum, i.e. the Holevo capacity, to a later time. In doing so, we first define

$$H(T; \rho) := \max_{\mathrm{av}[\mathcal{E}]=\rho} \chi(T(\mathcal{E})) \tag{6}$$

so that the maximum is to compute with respect to all ensembles having ρ as its average density. We get a function, depending on ρ (and T) only. If T is a partial trace onto a subalgebra, (6) has been called *entropy of ρ with respect to the subalgebra* by Connes *et al.* (1987; see also Narnhofer and Thirring, 1985). In this sense (6) could be called *entropy of ρ with respect to T.*

Schumacher and Westmoreland (1999) relate (6) to the efficiency of quantum channels. They denote the quantity in question by $\chi^*(\rho)$ and they call an ensemble saturation (6) an *optimal signal ensemble.*

It is important for our purposes to rewrite (6) by the help of (3) as

$$H(T;\rho) = S(T(\rho)) - E(T;\rho) \tag{7}$$

where E is an entanglement measure given by

$$E(T;\rho) = \min_{\sum p_k \rho_k = \rho} \sum p_j S(T(\rho_j)) \tag{8}$$

and the minimum is to compute over all convex decompositions of ρ. If T is a partial trace in a bipartite quantum system, (8) is the *entanglement of formation* introduced in Bennett *et al.* (1996), followed by the remarkable papers of Wootters (1997), and of Terhal and Vollbrecht (2000). Other examples are in Benatti *et al.* (1996, 1999). In Benatti *et al.* (2002), some relations between the quantity (8) for quite different channel maps are pointed out by the same authors.

It is well known that $E(T;\rho)$ is convex in ρ. Indeed, E is written as the convex hull of the function $S(T(\rho))$ in (8). The concavity of the entropy allows to perform (8) over the pure decompositions of ρ without changing the outcome of the minimization. Thus

$$E(T;\rho) = \min \sum p_j S(T(\pi_j)), \qquad \rho = \sum p_j \pi_j, \tag{9}$$

where all the π_k are pure and the minimum is to perform with respect to all pure decompositions of ρ. An ensemble of pure states saturating (9) will be called an *optimal ensemble.*

One should also notice the following. The equality between the two variational problems (8) and (9) is due to the concavity of $S(T(\rho))$. If we perform similar computations, however with a function not being concave, the two problems are essentially different. For certain channels, including all rank two and length two 1-qubit channels, the minimization (9) will be solved in the present paper.

The next section is a short introduction to the roof concept. We shall discuss how to estimate (9) from below by convex functions and from above by roofs.

Next we demonstrate the procedure for the phase-damping channels, where most things are now well understood. It follows the computation of concurrences and of E for some rank 2 channels. We give quite explicit computations to see the dependence of concurrences, foliations, and optimal ensembles from the Kraus operators.

A quite interesting observation is the following: Let us call *Kraus module* of T the linear span of the Kraus operators of T. Choi (1975) has shown that irreducibility of trace preserving cp-maps is a sole property of the Kraus module. Here we prove for length 2 and rank 2 channel maps that their optimal ensembles and, hence, their foliations are equal if they belong to the same Kraus module. It is not probable that this remarkable feature will survive for more complex channels. Nevertheless it seems worthwhile to ask for similarities of channels with identical Kraus modules.

It is a further consequence from our calculations that the foliations deform continuously by changing the entries of the Kraus operators for the class of channels considered. The foliations are more coarse properties of rank 2 cp-maps than concurrences.

2. ROOFS

There are some general features in the optimization problems we are interested in. They constitute the ground floor for more refined investigations.

Let us abstract from the specific values given at the pure states in (9) and let us start with an arbitrary real valued and continuous function $g(\pi)$ on the set of pure states. Assume, we like to extend g to a function defined for all density matrices. Clearly, there are many and quite arbitrary solutions for the problem. Let us denote by G one of these extensions.

To place a first restriction, we require the extension to be "as linear as possible." The requirement can be made precise as follows. Let ρ be a density operator. If there exists a pure decomposition of ρ such that

$$G(\rho) = \sum p_j g(\pi_j), \qquad \rho = \sum p_j \pi_j, \qquad (10)$$

we call the decomposition ρ-*optimal* or simply *optimal* for G. We also call a pure ensemble

$$\mathcal{E} = \{\pi_1, \ldots, \pi_m; \; p_1, \ldots, p_m\}, \qquad p_j > 0,$$

G-optimal if

$$G\left(\sum p_j \pi_j\right) = \sum p_j G(\pi_j).$$

A function, G, which allows an optimal decomposition (10) for every state ρ, I call a *roof* or, more literally, *a roof extension of g*. Roof extensions reflect the convex structure of the state space and they are, in a well-defined way, "as linear as possible." It is not easy to gain good examples of roof extensions in higher dimensions. In general, however, there are a lot of them for a given function g.

Let us now consider two further possibilities to extend g from the pure ones to all states: We may require the extension to be either concave or convex. In this

spirit we call a function on the state space a convex extension of g if the extension is a convex function. Similarly we speak of a *concave extension of g* if the extension is concave.

Lemma 1. *Given a convex, a concave, and a roof extensions of g. Then*

$$G_{\text{convex}}(\rho) \leq G_{\text{roof}}(\rho) \leq G_{\text{concave}}(\rho) \qquad (11)$$

for all density operators ρ.

The proof is almost trivial: One chooses a ρ-optimal decomposition (10). Then the very definition of a convex (concave) function establishes (11).

It follows that, given g, there can be only one convex roof extension, "the" *convex roof* determined by g. The family of roof extensions of g has just one member in common with the family of its convex extensions. Similarly there is just one *concave roof* which extends g.

As a matter of fact (Uhlmann, 1998), the convex roof with values $g(\pi)$ at the pure states is nothing than the solution of the variational problem which mimics (9).

$$G_{\text{convex,roof}}(\rho) = \min \sum p_j g(\pi_j), \qquad \rho = \sum p_j \pi_j. \qquad (12)$$

Here one has to run through all pure state decompositions of ρ. We get (9) by setting $g(\pi) = S(T(\pi))$ for pure π. On the other hand, if we take the maximum in (12) instead of the minimum, we obtain the concave roof extension of g.

Let us return for a moment to the more specific way of evaluating (9). To calculate $E(T; p)$ amounts to construct the convex roof with the function $g(\pi) = S(T(\pi))$. Looking at the roof property, there are some typical questions one should ask. For instance we may start by a set of parameterized mappings, T_s, and we like to know whether they have, perhaps for some ρ, identical optimal decompositions. More literally, we ask for a pure decomposition of ρ which is optimal for every T_s in an appropriate range of the parameter s. In the next sections we like to convince the reader that this point of view is quite fruitful. To do so, a remarkable property of convex and concave roofs is to be explained.

Let G be a convex roof on the density operators of a Hilbert space of finite dimension d. At first we use convexity: Let us fix a density operator ρ. There is at least one Hermitian operator, say Y^ρ, such that for all density operators ω

$$G(\rho) = \text{Tr} \, Y^\rho \rho, \qquad G(\omega) \geq \text{Tr} \, T^\rho \omega \qquad (13)$$

is valid. Now let us apply the roof property: There is a pure decomposition

$$\rho = \sum p_j \pi_j, \qquad p_j > 0, \quad \pi_j \text{ pure,}$$

which is G-optimal. Thus

$$\sum p_j \text{Tr} \, Y^\rho \pi_j = \text{Tr} \, Y^\rho \rho = G(\rho) = \sum p_j G(\pi_j).$$

Because of the inequality (13) this can hold if and only if

$$G(\pi_j) = \operatorname{Tr} Y^\rho \pi_j, \qquad j = 1, 2, \ldots$$

Let us now look at the convex set

$$\{\omega : G(\omega) = \operatorname{Tr} Y^\rho \omega\}. \tag{14}$$

By the very construction, G is convexly linear (affine) if restricted to this set. On the other hand, it contains ρ and, by the reasoning above, it contains every pure state which belongs to a G-optimal ensemble with average density ρ. Or, in other words, (14) contains all the pure states which can appear in an optimal decomposition of p. Let us collect all that in a theorem (Benatti *et al.*, 1996; Uhlmann, 1998).

Theorem 2. *Let g be a continuous function on the pure states.*

   (i)   There exists exactly one convex roof extension, G, of g.
   (ii) G can be represented by the optimization procedure (12).
   (iii) There exist optimal pure decompositions for every density operator ρ.
   (iv) Given ρ, G is convexly linear on the convex hull of all those pure states which can appear in an optimal decomposition of ρ.

Let us call the convex hull of all pure states appearing in all possible optimal pure decompositions of ρ the *optimal convex leaf of ρ*. As proved above, and stated in the theorem, G must be convexly linear on every optimal convex leaf.

Remark 2.   The remark concerns property (iii). The compact convex set of density operators enjoys a peculiarity: The set of their extremal points, i.e. the set of pure states, is a compact one. This allows to prove the existence of optimal decompositions (10) by the continuity of g. Then, by a theorem due to Caratheodory, one deduces the existence of optimal decompositions with a length not exceeding d^2, d the dimension of the Hilbert space which carries the density operators. It should be noticed that the compactness of the extermal points is an extra property. Counter examples are by no means exotic as seen by the set of trace preserving cp-maps. To get the conclusion (iii), it suffices for g to be continuously extendable to the closure of the set of extremal states.

Remark 3.   If G is a convex roof and G is the sum of two convex functions $G = G_1 + G_2$, then G_1 and G_2 are convex roofs, and every G-optimal ensemble is optimal for G_1 and G_2.

We need one more definition: We call a convex roof, G, *flat* if it allows for every ρ an optimal pure decomposition

$$\rho = \sum p_j \pi_j, \qquad G(\rho) \sum p_j G(\pi_j),$$

such that

$$G(\pi_1) = G(\pi_2) = \ldots = G(\pi_j) = \ldots$$

If this takes place, every ρ is contained in a convex subset, generated by pure states, on which the roof is not only convexly linear but even constant. The merit of flat roof, say G, is in the nice property that every function of G, say $f(G)$, is again a roof.

As a matter of fact, the convex roofs we are considering in the following enjoy even a stronger property: They are constant on the convex leaf of every ρ, i.e. g is constant on the pure states of *every* pure ensemble which is optimal for G.

3. PHASE-DAMPING CHANNELS

Let us consider some particularly simple examples of 1-qubit channels, the family of phase-damping channels to see what is going on in applying concurrences (Wootters, 1997), to 1-qubit channels according to Uhlmann (2001). For the most symmetric channel of the family, with $z = 0$ in (15), the theorem below and the insight into the foliation of the state space are due to Levitin (1994).

Let $|z| < 1$ be a complex number. Define the map T_z by

$$X = \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} \mapsto T_z(X) = \begin{pmatrix} x_{00} & zx_{01} \\ x^*x_{10} & x_{11} \end{pmatrix}. \tag{15}$$

The application of such a map does not change the pure states $|0\rangle \langle 01|$ and $|1\rangle\langle 1|$, and there are no other trace preserving and completely positive 1-qubit maps with this property than those given by (15).

Before starting the calculation, let us have a look on a bundle of parallel lines which foliates the state space. Geometrically, the 1-qubit state space can be represented by the Bloch ball. The Bloch ball is the unit ball sitting in the "Bloch space," that is in the real Euclidean 3-space of all Hermitian matrices of trace one. The "Pauli coordinates," x_j, of a matrix are read off from

$$X = \frac{1}{2}(x_0\mathbf{1} + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3).$$

With $x_0 = 1$ the real coordinates x_1, x_2, x_3, parameterize the Bloch space, and in this context they are referred to as "Bloch coordinates." Finally, the Bloch ball is the unit ball with respect to the Bloch coordinates.

Let us return to the phase-damping channels. The line $x_1 = x_2 = 0$, i.e. the x_3-axis, remains pointwise fixed under the mappings (15). On the intersection of the line with the Bloch ball $E(T_z; \rho)$ must be zero. One aim is to show that $E(T_z; \rho)$ is constant on the intersection of the Bloch ball with every line which is parallel to the line $x_1 = x_2 = 0$. These lines are given by fixing the values of x_1 and x_2 and letting x_3 free. Equivalently, such a line can be given by fixing x_{01} in (15). Now

we can write down a very simple convex roof, the restriction of the function

$$X \mapsto \sqrt{x_1^2 + x_2^2} \equiv 2|x_{01}|$$

onto the state space. Indeed, this function is convex on the Bloch space. It is even a seminorm there. And it is, trivially, a flat roof: It is constant on every line parallel to the x_3-axis of the Bloch space. The intersection of a given line with the Bloch ball is either empty, or touching the ball in just one point, or is a line segment with two pure states as end points. In the latter case, the pure states of the segment are

$$\begin{pmatrix} 1 - p & x_{01} \\ x_{10} & p \end{pmatrix}, \quad \begin{pmatrix} p & x_{01} \\ x_{10} & 1 - p \end{pmatrix}, \quad p(1 - p) = |x_{01}|^2, \tag{16}$$

with $2|x_{01}| < 1$. For $2|x_{01}| = 1$ the line touches the Bloch ball at one pure state.

With a flat roof, one can build other roofs just by taking a function of it: If f is a real function on the unit interval, $f(|x_{01}|)$ is again a flat roof. By an appropriate choice of f we shall find the form of $E(T_z; \rho)$.

To do so, we need to compute the determinant

$$\det T_z(X) = x_{00}x_{11} - zz^*x_{01}x_{10} = \det X + (1 - zz^*)x_{01}x_{10}.$$

Taking X pure, say $x_{jk} = a_j a_k^*$, only the second term is different from zero and we remain with

$$\det T_z(X) = (1 - |z|^2)|a_0 a_1|^2, \qquad x_{jk} = a_j a_k^*. \tag{17}$$

Using ideas from Wootters (1997) and Uhlmann (2000), we define the *concurrence* of T_z for Hermitian X by

$$C(T_z; X) := \sqrt{(1 - |z|^2)(x_1^2 + x_2^2)} \tag{18}$$

The definition differs from the one used in Uhlmann (2001) by a factor of 2. It influences, here and later on, the appearance of some equations. The concurrence is a seminorm on the Bloch ball, and, if restricted to the Bloch ball, it is the unique convex roof satisfying

$$C(T_z; \pi) = 2\sqrt{\det T_z(\pi)}, \qquad \pi \text{ pure} \tag{19}$$

Following again Bennett *et al.* (1996) and Wootters (1997), we introduce

$$h(x) = -x \log x - (1 - x) \log(1 - x),$$

and, using ad hoc notations,

$$h_1(x) = h\left(\frac{1 + x}{2}\right)$$

and

$$h_2(x) = h_1(y), \qquad y = \sqrt{1 - x^2}.$$

Theorem 3. *For all $|z| < 1$ and all density operators ρ*

$$E(T_z; \rho) = h_2(C(T_z; \rho)) \tag{20}$$

holds. It is $E(\rho_1) = E(\rho_2)$ for two density operators, ρ_1 and ρ_2, if and only if they have equal distances to the x_3-axis of the Bloch space. The pairs of optimal pure states are given by (16).

Proof: We already know that (20) is a roof with the desired values at the pure states. We only have to show that it is convex. Then, by the uniqueness theorem, we are done. One calculates the first and the second derivatives of h_2, assuming $\log \equiv \ln$. At first we get

$$h_2'(x) = \frac{x}{2y} \ln \frac{1+y}{1-y}, \qquad n_3''(x) = \frac{1}{2y^3} \ln \frac{1+y}{1-y} - \frac{1}{y}.$$

For $x \geq 0$ and $y \geq 0$ we find $h_2' \geq 0$. Expanding the logarithm, one gets

$$h_2''(x) = -\frac{1}{y} + \frac{1}{y^3}\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right) \geq 0$$

within $0 < y < 1$, which proves the convexity of h_2. Let $C(\rho)$ be a convex function on the state space (or on another convex set) with values between 0 and 1. Let us denote by a dot the differentiation of C in an arbitrary direction. Then

$$h_2(C)\ddot{} = h_2''(C)\dot{C}\dot{C} + h_2'(C)\ddot{C}.$$

By the convexity of C one gets $\ddot{C} \geq 0$, and we have seen $h' \geq 0$, on the unit interval. Thus, the second term is not negative. As we know $h'' \geq 0$, we have shown the convexity of the function (20), and we done. $\square$

Lemma 4. *Let C be a convex function with values in the unit interval, defined on a finite dimensional convex set. Then $h_2(C)$ is convex.*

It is remarkable that the whole set of phase-damping channels induces a single foliation of the state space: The foliation is a property of the Kraus module belonging to the channels. The single foliation forces the concurrences to differ by a factor only if z is changing.

Now we have to add a further structural element as a guide in treating more general 1-qubit channels. What we have in mind is, up to a contracting factor, a reflection on the plane $z_3 = 0$. In Bloch space a reflection is not a proper rotation. Its functional determinant must be negative, enforcing its implementation by an antilinear operator in the 1-qubit space. The antilinearity is unavoidable.

Let us define an antilinear operator ϑ_z by

$$\vartheta_z \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \sqrt{1 - zz^*} \begin{pmatrix} a_1^* \\ a_0^* \end{pmatrix}. \tag{21}$$

The operator is an Hermitian one; i.e., it satisfies

$$\langle \phi_1 | \vartheta_z | \phi_2 \rangle = \langle \phi_2 | \vartheta_z | \phi_1 \rangle$$

for all pairs of vectors. Being antilinear, ϑ acts to the right, but not to the left. With two arbitrary vectors,

$$|a\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \qquad |b\rangle = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix},$$

we get the following relation.

$$\langle a | \vartheta_z | b \rangle = \sqrt{1 - zz^*}(a_0 b_1 + a_1 b_0)^*. \tag{22}$$

In combination with (17) we obtain the equation

$$4 \det T_z(|a\rangle\langle a|) = |\langle a|\vartheta_z|a\rangle|^2, \tag{23}$$

which, together with (19), can be written

$$C(T_z; |a\rangle\langle a|) = |\rangle a|\vartheta_z|a\rangle|. \tag{24}$$

Because the concurrence is a convex roof, we can now return to one of the properties of such function:

$$C(T_z; \rho) = \min \sum p_j |\langle \phi_j | \vartheta_z | \phi_j \rangle| \tag{25}$$

where the minimum runs through all possible ways of representing ρ as a convex combination

$$\rho = \sum p_j |\phi_j\rangle\langle \phi_j|.$$

The minimum will be attained by choosing the decomposition with a pair (16) of pure states. With the same optimal decomposition of ρ the optimization problem for $E(T_z; \rho)$ can be saturated. (The diagonal entries of ρ must be 1/2.)

Remark 4. Let us assume we like to solve for the phase-damping channels the variational problems (8), but we like to replace the minimization by the maximization, resulting in concave roofs. Now the foliation for the maximization is given by the intersection of the planes perpendicular to the x_3-axis with the Bloch ball. The foliation is the same for all z. The intersection of a plane with a ball contains a whole circle of extremal states. Hence there are very many different optimal decompositions for given mixed ρ.

4. CONCURRENCE

Let us now discuss some generalities for rank 2 channels and let us see where the difficulties are. A quantum channel, T, is of rank k if the maximal rank of all pictures, $T(\rho)$, is k.

Let T be of rank 2. We assume in addition that T maps into a 1-qubit space. Then the determinant of $T(\rho)$ is the product of the eigenvalues of $T(\rho)$. We define the concurrence, $C(T;\rho)$, of T to be the convex roof which attains at pure states the values

$$C(T;\pi) = 2\sqrt{\det T_z(\pi)}, \qquad \pi \text{ pure}, \tag{26}$$

completely similar to (19). Being convex with values between 0 and 1, we can literally repeat the construction as in (20). The result is a convex function which coincides at pure states with E. But in general we do not know whether (26) is a *flat* roof. Hence, we only can conclude

$$E(T;\rho) \geq h_2(C(T;\rho)). \tag{27}$$

If, however, (26) is a flat roof, then the right-hand side of (26) is a flat convex roof and we have two convex roofs agreeing on pure states. Then equality must hold.

Lemma 5. *If C as given by (26) is a convex roof, then we have*

$$C \text{ flat} \implies E(T;\rho) = h_2(C(T;\rho)). \tag{28}$$

It is useful to know whether the concurrence of a channel map T is a flat roof. The following theorem gives a whole class of them.

Theorem 6. *Let ϑ be an antilinear Hermitian operator and define C_ϑ as the convex roof extension of*

$$C_\vartheta(|\phi\rangle\langle\phi|) = |\langle\phi|\vartheta\rangle| \tag{29}$$

for all pure states π. Then the convex roof C_ϑ is flat. It is

$$C_\vartheta(\rho) = \max\left\{0, \lambda_1 - \sum_{j>1}\lambda_j\right\}, \tag{30}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots$ are the ordered eigenvalues of

$$(\rho^{1/2}\vartheta\rho\vartheta\rho^{1/2})^{1/2}.$$

Corollary 1. *If there exists an antilinear Hermitian ϑ such that*

$$4\det T(\pi) = \operatorname{Tr}\pi\vartheta\pi\vartheta, \qquad \pi \text{ pure}, \tag{31}$$

is valid for all pure density operators then

$$C_\vartheta(\rho) = C(T; \rho) \tag{32}$$

is a flat roof and (28) is valid.

This theorem is proved in Uhlmann (2000). It provides a certain application of the methods of Wootters and others (see Wootters, 2002). As already mentioned, with T the partial trace, $E(T; \rho)$ is the entanglement of formation. In the 2-qubit system ϑ is the Hill–Wootters conjugation.

For 1-qubit maps (30) reads $\lambda_1 - \lambda_2$ and one can become more explicit. Abbreviate

$$\xi = \left(\rho_1^{1/2} \rho_2 \rho_2^{1/2}\right)^{1/2}.$$

Taking the trace of the characteristic equation of ξ results in

$$\operatorname{Tr} \xi^2 + 2 \det \xi = (\operatorname{Tr} \xi)^2$$

and the squared sum of the eigenvalues of ξ becomes

$$(\lambda_1 + \lambda_2)^2 = \operatorname{Tr}(\rho_1 \rho_2) + 2 \det \xi.$$

Combined with the relation

$$(\lambda_1 - \lambda_2)^2 = (\lambda_1 + \lambda_2)^2 - 4 \det \xi,$$

it yields

$$(\lambda_1 - \lambda_2)^2 = \operatorname{Tr}(\rho_1 \rho_2) - 2\sqrt{(\det \rho_1)(\det \rho_2)}.$$

Substituting $\rho_1 = \rho$ and $\rho_2 = \vartheta \rho \vartheta$ provides (Uhlmann, 2000),

$$C_\vartheta(\rho)^2 = \operatorname{Tr}(\rho \vartheta \rho \vartheta) - 2(\det \rho) \det \sqrt{\vartheta^2}. \tag{33}$$

5. 1-QUBIT CHANNELS OF LENGTH 2

In this section we like to show that the corollary to the preceding theorem applies to 1-qubit channels of length 2. As we shall see, the existence of an antilinear Hermitian θ fulfilling (31) does not depend on trace preserving.

Let A and B be two linear independent operators on a two-dimensional Hilbert space and

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix},$$

their matrix representations with respect to a reference basis.

$$T(X) = AXA^\dagger + BXB^\dagger \tag{34}$$

is a completely positive map of length 2. The following has been proved in Uhlmann (2001) by straightforward computation.

Theorem 7. *There is an antilinear Hermitian operator $\vartheta_{A,B}$ such that for all pure states*

$$4 \det T(|a\rangle\langle a|) = |\langle a|\vartheta_{A,B}|b\rangle|^2,$$

where T is given by (34).

If such an operator exists, it is determined by T up to a phase factor only. The ambiguity is a natural one due to a geometric phase.

To describe ϑ, one can introduce its matrix representation

$$\vartheta_{A,B}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \alpha_{00}a_0^* + \alpha_{01}a_1^* \\ \alpha_{10}a_0^* + \alpha_{11}a_1^* \end{pmatrix} \tag{35}$$

and express the matrix elements by those of A and B.

$$\alpha_{00} = 2(b_{10}a_{00} - a_{10}b_{00})^*, \quad \alpha_{11} = 2(a_{01}b_{11} - b_{01}a_{11})^*, \tag{36}$$

$$\alpha_{01} = \alpha_{10} = (a_{00}b_{11} - a_{11}b_{00} + a_{01}b_{10} - a_{10}b_{01})^*. \tag{37}$$

Up to a factor, because of another normalization of the concurrence, this is in agreement with Uhlmann (2001). Clearly, ϑ must be skew-symmetric in the matrix entries of the Krause operators,

$$\vartheta_{A,B} + \vartheta_{B,A} = 0.$$

Comparing the equations above with phase-damping channels, one gets $\vartheta_{A,B} = -\vartheta_z$.

One may expect a more transparent representation than (36) and (37). This is possible by the spin-flip operator θ_f, the "fermion conjugation" for 1 qubit,

$$\theta_f\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} a_1^* \\ -a_0^* \end{pmatrix}.$$

After calculating

$$\begin{pmatrix} (b_{10}a_{00} - a_{10}b_{00})^* & (a_{00}b_{11} - a_{10}b_{01})^* \\ (a_{01}b_{10} - a_{11}b_{00})^* & (a_{01}b_{11} - a_{11}b_{01})^* \end{pmatrix}$$

and comparing that expression with (36) and (37), we get

$$\vartheta_{A,B} = A^\dagger\theta_f B - B^\dagger\theta_f A. \tag{38}$$

Remember $\theta_f^\dagger = -\theta_f = \theta_f^{-1}$ to see that (38) is an antilinear Hermitian operator—as it should be. Now assume a transformation of the Kraus operators according to

$$A, B \to C_1 A C_2, C_1 B C_2. \tag{39}$$

Because of

$$C_1^\dagger \theta_f C_1 = (\det C_1)^* \theta_f$$

the antilinear operator must change as follows:

$$\vartheta_{A,B} \to (\det C_1)^* C_2^\dagger \vartheta_{A,B} C_2. \tag{40}$$

That it is of value to classify superoperators up to a transformation (39) is also seen by the results of Verstraete and Verschelde (2002).

Let T' be another length 2 cp-map with Kraus coefficients A' and B', and let us assume a linear dependence

$$A' = \mu_{00} A + \mu_{01} B, \qquad B' = \mu_{10} A + \mu_{11} B. \tag{41}$$

By the help of (38), or by observing that (36) and (37) can be expressed by determinants in the coefficients of the Kraus operators, one can reproduce the relation

$$\vartheta_{A',B'} = (\mu_{00}\mu_{11} - \mu_{01}\mu_{10})^* \vartheta_{A,B}. \tag{42}$$

Why is this interesting? It shows that our procedure associates, up to a scalar factor, to every pair of operators, chosen from the linear span of A and B, the same antilinear operator. With other words, to every two-dimensional Kraus module a one-dimensional linear space of antilinear Hermitian operators is attached.

Remark 5. Regard the Kraus modules for the 1-qubit channels as the points of the second Grassmann manifold of the space of linear operators. By attaching the multiples of $\vartheta_{A,B}$ to the corresponding points, one gets the line bundle. It is dual to the determinant bundle as one can deduce from what follows. For the time being, we shall not follow further this way.

An observation, related to (42), is

$$A' \otimes B' - B' \otimes A' = (\mu_{00}\mu_{11} - \mu_{01}\mu_{10})(A \otimes B - B \otimes A). \tag{43}$$

If we apply the operator identity (43) to a $|aa\rangle$, we get an antisymmetric 2-qubit vector. There is, up to a factor, only one such vector and the yet-unknown factor must transform as in (43). Performing the calculations, one gets

$$(A \otimes B - B \otimes A)|aa\rangle = \frac{1}{2}\langle a|\vartheta_{A,B}|a\rangle^*(|01\rangle - |10\rangle). \tag{44}$$

Remember that for a single channel, T, only the absolute value of the expectation value at the right-hand side is relevant.

Two channels may be called "unitary equivalent" if

$$T'(\rho) = U_1 T \left(U_2^{-1} \rho U_2 \right) U_1^{-1}$$

with two unitaries U_1 and U_2. For trace preserving channels we can assume

$$A = \begin{pmatrix} a_{00} & 0 \\ 0 & a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_{01} \\ b_{10} & 0 \end{pmatrix} \tag{45}$$

up to unitary equivalence, (Ruskai *et al.*, 2001). Then ϑ becomes diagonal,

$$\vartheta_{A,B} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 2 \begin{pmatrix} (b_{10}a_{00}a_0)^* c \\ (-b_{01}a_{11}a_1)^* c \end{pmatrix}. \tag{46}$$

Abbreviating

$$y_0 = 2b_{10}a_{00}, \qquad y_1 2b_{01}a_{11}$$

we can write

$$\vartheta \rho \vartheta = \begin{pmatrix} \rho_{00} y_0 y_0^* & -\rho_{10} y_0 y_1^* \\ -\rho_{01} y_1 y_0^* & \rho_{11} y_1 y_1^* \end{pmatrix}.$$

It follows

$$\mathrm{Tr}\, \rho \vartheta \rho \vartheta = \rho_{00}^2 y_0 y_0^* - \rho_{01}^2 y_0 y_1^* - \rho_{10}^2 y_1 y_0^* + \rho_{00}^2 y_1 y_1^*.$$

On the other hand,

$$2 \det \rho \det \sqrt{\vartheta^2} = 2|y_0 y_1|(\rho_{00}\rho_{11} - \rho_{01}\rho_{10}).$$

Pasting all things together, we get from (33)

$$C^2 = (\rho_{00}|y_0| - \rho_{11}|y_1|)^2 + 2|y_0 y_1|\rho_{01}\rho_{10} - \rho_{01}^2 y_0 y_1^* - \rho_{10}^2 y_0^* y_1.$$

We now choose the square roots of $y_0 y_1^*$ and $y_0^* y_1$ such that their product becomes positive. Then we can write the remaining terms above as follows:

$$-\left(\rho_{01}\sqrt{y_0 y_1^*} - \rho_{10}\sqrt{y_0^* y_1} \right)^2$$

so that, finally, we see

$$C(T, \rho)^2 = 4L_1(\rho)^2 + 4L_2(\rho)^2 \tag{47}$$

were L_1 and L_2 are real-valued and linear in the entries of ρ,

$$L_1(\rho) = \rho_{00}|b_{10}a_{00}| - \rho_{11}|b_{01}a_{11}| \tag{48}$$

$$L_2(\rho) = i \left(\rho_{01}\sqrt{b_{10}b_{00}b_{01}^* a_{11}^*} - \rho_{10}\sqrt{b_{10}^* a_{00}^* b_{01} a_{11}} \right) \tag{49}$$

and we have to choose the signs of the roots according to

$$\sqrt{b_{10}a_{00}b_{01}^*a_{11}^*}\sqrt{b_{10}^*a_{00}^*b_{01}a_{11}} \geq 0. \tag{50}$$

The result compares well with the more symmetrical case of the phase-damping channels: C is the square root of a positive semidefinite quadratical form. Geometrically, the points of constant concurrence and, hence, of E are ellipse-based cylinders.

In the nondegenerate case none of the two linear forms vanish identically. The foliation of the state space at which the concurrence and also $E(T;\rho)$ remain constant are given by the straight lines which are the intersection of the two families of planes $L_1 = $ constant, $L_2 = $ constant, in the Bloch space. There is just one straight line at which the concurrence is zero. It goes through the two pure states which are mapped onto pure states by T. Up to normalization these two pure states belong to the vectors

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \qquad b_{10}a_{00}a_0^2 = b_{01}a_{11}a_1^2.$$

They are mapped by T to vector states of the form

$$\begin{pmatrix} a_0' \\ a_1' \end{pmatrix}, \qquad b_{10}a_{11}(a_0')^2 = b_{01}a_{00}(a_1')^2 = 0.$$

Let us now shortly look at the *degenerate case* in which $b_{01}b_{10}$ is zero in (45). The amplitude-damping channels are well-known examples. They can be defined by the Kraus operators

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{pmatrix} \tag{51}$$

with $0 < p < 1$. The action of T is described by

$$\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \mapsto \begin{pmatrix} \rho_{00} + (1-p)\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & p\rho_{11} \end{pmatrix}$$

(47) becomes

$$C(T;\rho) = 2\sqrt{p(1-p)}\rho_{11}. \tag{52}$$

The two families of planes degenerate to one family, the planes perpendicular to the 3-axis of the Bloch space. The foliation dictates the behavior of $C(T;\rho)$ and $E(T;\rho)$.

One observes that, given ρ_{11}, the entropy $S(T(\rho))$ becomes maximal if the off diagonal entries of ρ vanish. Therefore, if ρ' is the diagonal part of ρ, We get

$$E(T;\rho) = E(T;\rho'), \qquad H(T;\rho) \leq H(T;\rho')$$

and, to obtain the Holevo capacity, it suffices to consider diagonal density operators only:

$$C^{(1)} = \max H(T; \rho'), \qquad \rho' \text{diagonal.}$$

Writing r for ρ_{11}, such that $0 \leq r \leq 1$, we can rewrite the capacity as follows:

$$C^{(1)}(T) = \max_{0 \leq r \leq 1} \left[h(pr) - h\left(\frac{1 - \sqrt{1 - 4p(1-p)r^2}}{2} \right) \right]. \qquad (53)$$

Because $H(T; \rho)$ is a concave function, (53) is a concave function of r and, obviously, not degenerate. Therefore, for any given $0 < p < 1$, there is exactly one value r_0 of r at which the maximum in (53) is attained.

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